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# An Exact Analytical Solution for Gaseous Flow and Heat Transfer in Microtubes with Constant Wall Temperature

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**ABSTRACT**

It is known that slip flow and temperature jump phenomena play a significant role in micro scale investigations. In this paper, exact analytical solutions for the flow and the convective heat transfer of gaseous flow passing through microtubes are derived for the first time in form of Whittaker function. Here, it is assumed that both flow and heat transfer is fully developed in a microtube with constant wall temperature. The solution is obtained by considering the Navier-slip conditions for flow and heat transfer at walls. Here, a modal analysis technique is employed to achieve possible solutions of this scenario. Due to the eigenvalue form of governing equations, obtaining the closed form exact solution for this problem is too difficult from mathematical point of view and previous studies have been restricted to numerical and approximate series expansion solutions. In this study, an additional constraint is introduced using the definition of the mean temperature and employed to obtain possible eigenvalues related to this problem. Finally, by implementing a scaling law of the Nusselt number of laminar flow in closed conduits, an exact analytical solution for temperature distribution and the heat transfer are derived. It was found that increasing the Prandtl number increases the Nusselt number and increasing the Knudson number decreases the Nusselt number. Based on the obtained solution, the effect of Prandtl number and Knudsen number on heat convection of microtubes is studied in detail.

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## 1. Introduction

During last decades, increasing attention has been allocated to mini and micro scale investigations stemming from high performance small-scale equipment in engineering and industry fields such as bio-related applications, medical diagnosis, refrigerant systems, chemical reactors and electronic component cooling systems. Previous studies which have been carried out on the fluid flow in micro scale geometries clearly revealed that the flow and the heat transfer characteristics are different in comparison with conventional macro scale flows. These characteristics of the flow are mainly related to slip situations appearing around the contact surface of a fluid and a solid which generally shows a direct relation with the dimensionless parameter Knudsen number. This dimensionless number is defined as the ratio of the mean-free molecular distance to the system macro length scale. The molecular distance in gases is much greater than in liquids so the Knudsen numbers in gases and liquids are generally categorized in the ranges of $0.001<Kn<0.1$ and $Kn<0.001$, respectively, showing the concept that molecular distance in gases is greater than in liquids. It is pertinent to mention that while the Knudsen number obtains its value in range of $0.001<Kn<0.1$ the continuum equations such as Navier-Stokes and Energy conservative equations are still applicable but they must be solved subject to the velocity slip and the temperature jump conditions around the contact surface of the fluid and the

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solid.

There are many seminal works that have been focused on the non-slip situation for the flow and the heat transfer passing through conduits employing theoretical methods. Shah [1] investigated possible effects of the cross-section shape on the heat transfer inside plane conduits. He showed that in a constant heat flux situation making rounded the corner of the cross section can slightly influence the heat transfer. In a following research, Shah and London [2] studied effects of curvature using the finite difference method for fully developed conditions and showed the centrifugal force arising from the curvature increases the heat transfer. Ou *et al.* [3] and Hieber [4] studied the natural heat convection/transfer in isothermal pipes both in cooling and heating situations in large Prandtl number fluids by implementing the numerical method for a horizontal tube.

In recent decades due to the increasing request in high performance small-scale equipment, some researchers concentrated their investigations on laminar slip flows with the temperature jump as an important phenomenon. Among these studies, Hettiarachchi *et al.* [5] used the finite volume method to investigate a three-dimensional laminar slip flow and its heat transfer in rectangular cross sections. They presented a correlation for the fully developed friction factor as a function of the Knudsen number and the aspect ratio. The obtained results indicate that the velocity slip increases the Nusslet number while the temperature jump decreases it. In a following research, Lee and Garimella [6] extended the problem to investigate effects of the entrance region of the rectangular micro-channel employing a 3-dimensional numerical method. Moreover, they investigated different cross sections in the fully developed laminar flow and obtained both the local and the average values of the Nusselt numbers as a function of dimensionless axial distance and channel aspect ratio. Furthermore, Renksizbulut *et al.* [7] investigated the flow and the heat transfer in the entrance region of rectangular micro-channels numerically with the slip flow boundary condition using a control-volume method.

Beside numerical methods, there are some empirical studies which can be more reliable and are generally used to validate solutions achieved by numerical methods [8]. As an example for empirical methods in the pipe, Sieder and Tate [9] reported an accurate correlation between the ratio of viscosity of the fluid stream to the viscosity of the fluid beside the wall with the heat transfer coefficient both in heating and cooling conditions. Morgan [10], Churchill and Chu [11] and Whitaker [12] derived experimental correlations that mainly correspond to the area and the time-averaged Nusselt number. Mori and Futagami [13] visualized the pattern of secondary flows arises from the buoyancy force to study and analyze effects of these phenomena on the heat convection of the laminar flow regime.

In recent years, there has been an increasing attention in experimental investigations to achieve non-dimensional correlations between variables and the flow fields and the heat transfer in micro-channels. These correlations have an important value in designing, constructing and controlling high efficient heating and cooling equipment. Colin *et al.* [14], investigated the slip flow situation using a second-order boundary condition in rectangular ducts analytically. In the second part of the investigation, they employed an experimental approach to show the accuracy of the presented solution. Hetsroni *et al.* [15,16] presented experimental and theoretical analysis for the fluid flow and the heat transfer in micro-channels. They presented results relating to small Knudsen number cases in circular, triangular, rectangular and trapezoidal micro ducts. The effect of the geometry and the axial heat flux on the energy dissipation is investigated. Furthermore, Chen et al. [17] performed an experimental study to investigate characteristics of the fluid flow and the heat transfer in a micro-channel heat sink with different hydraulic diameters ranging from $57\mu m$ to $267\mu m$. They found that the surface roughness, the viscosity and the channel geometry are the most effective parameters on flow characteristics in micro scale channels.

In the recent years, Shahmardan *et al.* [18] achieved exact analytical solution for the convective heat transfer in straight rectangular ducts and validated it for both H1 and H2 boundary conditions under constant heat flux at the duct wall. Norouzi and Davoodi [19] have derived an exact analytical solution for forced convective heat transfer inside straight pipes under an isothermal boundary condition using modal analysis of the eigenvalue form of the heat transfer equation. The main difference between current study and Norouzi and Davoodi work [18] is that the presented exact analytical solution in this paper accounts for the temperature jump and the velocity slip conditions at wall for gaseous fluid flow through microtubes. Unfortunately, finding the exact analytical solutions for a complex situation is not possible in most of times and it restricts into solutions of some simple scenarios. Therefore, some researchers use approximate analytical methods such as standard perturbation, homotopy perturbation methods, differential transform method and stochastic techniques. Among the studies that used approximate analytical approaches, Morton [20], Hanratty [21], Iqbal and Stachiewicz [22, 23] and Faris and Viskanta [24] used a series expansion method to show the effects of the free and the forced convection using the Rayleigh number as a perturbation parameter. A general model for predicting the pressure drop in micro-channels of an arbitrary cross section has been reported by Bahrami *et al.* [25]. Their results showed a good agreement with experimental and numerical data for a wide variety of situations. Khan and Yovanovich [26] studied the laminar forced convection in 2-dimensional rectangular micro and nano channels under both hydrodynamically and thermally fully developed conditions in the slip-flow regime. They solved the momentum and the energy equations with a first order velocity slip and a temperature jump conditions at the wall of the channel and presented a closed form solution for the Nusselt number in terms of effective parameters such as the Reynolds number, the Knudsen number and the Prandtl number. In recent years, Hooman *et al.* [27] obtained an analytical solution based on a perturbation method for fully developed flow in both circular and parallel plates. They studied the scaling effect of the variable property, the viscous dissipation, the velocity slip, and the temperature jump on the slip flow forced convection of gaseous flows. The slip flow heat transfer in annular micro-channels has been analyzed by Duan and Muzychka [28]. They found that for the slip flow cases, the Nusselt number decreases by increasing the Knudsen number and it is lower than those for the continuum flow. Yu and Ameel [29, 30] studied the laminar forced convection in a thermally developing slip flow for both constant wall temperature and heat flux boundary conditions using the integral transient technique. Laminar fully developed flows in mini and micro-channels for hyper-elliptical and regular polygonal cross sections have been studied analytically by Tamayol and Bahrami [37]. Shomali and Rahmati [32] developed Cascaded Lattice Boltzmann Method with second order slip boundary conditions to study gas flows in a microchannel in the slip and transition flow regimes with a wide range of Knudsen numbers. They consider the effect of wall confinement on the effective mean free path of the gas molecules using a function with nonconstant Bosanquet parameter instead of the constant one. Rahmati and Najati [38] investigated an incompressible thermal flow in a micro-Couette in the presence of a pressure gradient utilizing the analytical solution of the Burnett equations with first-order and second-order slip boundary conditions. Barik and Nayak [39] investigated the laminar forced convective heat transfer and fluid flow characteristics for Al2O3-water nanofluid flowing in different bend (i.e., 180o and 90o) pipes numerically in a three-dimensional computational domain using the finite volume technique. Tajik *et al*. [40] carried out an experimental study to investigate the effect of adding Al and Cu nanoparticles to the base fluid (water) on the heat transfer rate in a spirally coiled tube.

Usually exact analytical solutions are more reliable and can be used to check the accuracy of other methods but due to difficulties in solving the momentum and the energy equations, the analytical solutions are not always easy to obtain. As mentioned in the literature part, previous researches have mainly focused on numerical and series expansion analytical methods while in this study the momentum and energy equations have been solved directly. In this paper, a modal analysis technique is employed to achieve possible solutions of this scenario. Due to the eigenvalue form of governing equations, developing the closed form exact solution is too difficult and previous studies have been solved it numerically. In this study, an additional constraint is introduced using the definition of the mean temperature and employed to obtain possible eigenvalues related to this problem. In the current study, laminar fully developed flow and heat transfer in straight microtubes is investigated and an exact analytical solution for the velocity and temperature distributions corresponding to the temperature jump and the velocity slip conditions at wall for gaseous fluids are obtained in form of Whittaker function for the first time. Finally, the effect of Prandtl number and Knudsen number on heat convection of isothermal microtubes is studied in detail.

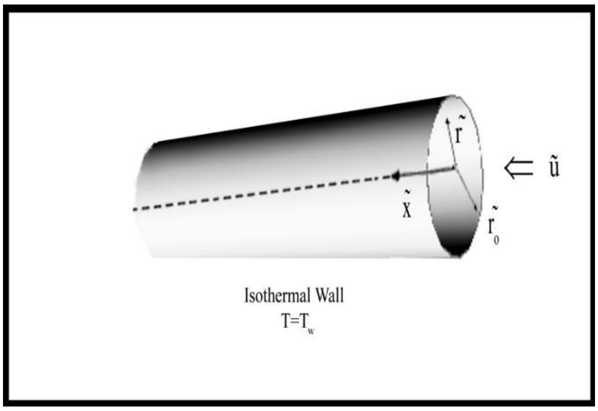


**Figure 1.** Geometry of microtube in current study.

## 2. Physics of Problem

The problem is modeled as a circular microtube as shown in Fig. 1. The flow assumes to be steady, two-dimensional, incompressible, and laminar. In plain pipes, due to the symmetry situation and the fully developed condition both $V_r$ and $V_\theta$ are equal to zero. The flow condition is considered as viscous fluid flow conditions where derivatives of all of the parameters of the velocity with respect to z-direction are eliminated except the pressure. In this study slip, assumption, and symmetry at the center of pipe are considered and, the viscous dissipation energy is neglecting and the constant temperature is considered as wall boundary condition.

## 3. Mathematics Modeling

To facilitate analytical solutions for differential equations, defining the viscous flow problem, it is pertinent to employ non-dimensional variables. The relevant dimensionless parameters involved in the current study are defined as following:

$$r = \frac{\tilde{r}}{r_o} \ . \ z = \frac{\tilde{z}}{r_o} \ . \ u = \frac{\tilde{u}}{U} \ . \ p = \frac{\tilde{p}.r_o}{\mu.U} \ . \\ Kn = \frac{\lambda}{r_o} \ . \ T = \frac{\tilde{T} - \tilde{T}_w}{\tilde{T}_m - \tilde{T}_w} \ . \ Nu = \frac{hD}{k_f} \qquad (1)$$

### *3.1. Flow solution*

The momentum and continuum equations can generally express as follows:

$$\nabla.\tilde{V} = 0 \qquad (2a)$$

$$\rho\tilde{V}.\nabla\tilde{V} = -\nabla\tilde{P} + \mu\nabla^2\tilde{V} \qquad (2b)$$

The momentum equation in z direction can be expressed as:

$$-\frac{\partial \tilde{p}}{\partial \tilde{z}}+\mu(\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\frac{\partial \tilde{u}}{\partial \tilde{r}}))=0 \qquad (3)$$

After substituting dimensionless parameter introduced in (1) into (3) we have:

$$-\frac{\partial p}{\partial z}+(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}))=0 \qquad (4)$$

In the fully developed conditions, the pressure gradient in the axial direction can be expressed as:

$$\frac{\partial P}{\partial z}=cte<0 \qquad (5)$$

Therefore,

$$\frac{\partial P}{\partial z}=-G \qquad (6)$$

where $G$ is a constant value that represents absolute axial pressure drop.

Considering slip assumption and symmetry at center of pipe, boundary conditions for solving the momentum equation may be presented as:

$$at\ \ \tilde{r}=0\rightarrow\frac{\partial \tilde{u}}{\partial \tilde{r}}=0 \qquad (7a)$$

$$at\ \ \tilde{r}=r_o\rightarrow \tilde{u}_s=\frac{F-2}{F}Kn\,r_o\left.\frac{d\tilde{u}}{d\tilde{r}}\right|_{\tilde{r}=r_o} \qquad (7b)$$

where $F$ and $Kn$ represent the tangential momentum accommodation coefficient and the Knudsen number, respectively. Substituting corresponding dimensionless parameters, presented in (1) into (7), and defining a slip velocity coefficient as$\beta_v$, the dimensionless form of slip boundary condition and symmetry constraint will be:

$$at\ \ r=0\rightarrow\frac{\partial u}{\partial r}=0 \qquad (8a)$$

$$at\ \ r=1\rightarrow u_s=-\beta_v\left.\frac{du}{dr}\right|_{r=1} \qquad (8b)$$

$$\beta_v=\frac{2-F}{F}Kn \qquad (8c)$$

Solving equation (4) respect to appropriate boundary condition (8) leads to:

$$u(r)=2\left(1-r^2+\frac{1}{2}\beta_v\right) \qquad (9)$$

Using Eqs. (1), (9), and (4), the non-dimensional axial pressure gradient may be presented as:

$$\frac{\partial\, p}{\partial\, z}=-8 \qquad (10)$$

## 3.2. Heat transfer solution

Neglecting the dissipation energy and assuming an incompressible fluid flow, the energy equation can be represented as:

$$\rho\, c_p\, \tilde{V}.\nabla\tilde{T}=k\nabla^2\tilde{T} \qquad (11)$$

Similar to the velocity field, due to symmetry situations, derivatives of all of the temperature variable with respect to $z$ is eliminated. In the fully developed thermal conditions, the energy conservative equation expresses as:

$$\rho C_p\tilde{u}\frac{\partial \tilde{T}}{\partial \tilde{z}}=\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(k_f\tilde{r}\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) \qquad (12)$$

The appropriate constraints are the temperature jump at the wall and the symmetry situation at the center of the pipe which is presented as:

$$at\ \ \tilde{r}=0\rightarrow\frac{\partial \tilde{T}}{\partial \tilde{r}}=0\ \ (\text{or T is limited}) \qquad (13\text{-a})$$

$$at\ \ \tilde{r}=r_o\rightarrow$$

$$\tilde{T}_s-\tilde{T}_w=\frac{F_t-2}{F_t}\frac{Kn}{Pr}\frac{2\gamma}{1+\gamma}r_o\left.\frac{d\tilde{T}}{d\tilde{r}}\right|_{\tilde{r}=r_o} \qquad (13\text{-b})$$

where $F_t$ is the energy accommodation coefficient, $\gamma$is the ratio of specific heats $(c_p/c_v)$, $Pr$is the Prandtl number. Thermal fully developed condition is defined as [19]:

$$\frac{\partial T}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\tilde{T}-\tilde{T}_w}{\tilde{T}_m-\tilde{T}_w}\right)=0 \qquad (14)$$

Thus, one can easily derive the thermal fully developed condition as:

$$\frac{\partial \tilde{T}}{\partial \tilde{x}}=\left(\frac{\tilde{T}-\tilde{T}_w}{\tilde{T}_m-\tilde{T}_w}\right)\frac{d\tilde{T}_m}{d\tilde{x}}=\tilde{T}\frac{d\tilde{T}_m}{d\tilde{x}} \qquad (15)$$

By applying the heat balance on a differential control volume, the axial mean temperature gradient can be obtained:

$$h(\tilde{T}_w-\tilde{T}_m)\tilde{p}d\tilde{x}=\rho A\tilde{U}c_p d\tilde{T}_m$$
$$\Rightarrow\frac{d\tilde{T}_m}{d\tilde{x}}=\frac{2h(\tilde{T}_w-\tilde{T}_m)}{\rho\tilde{r}Uc_p} \qquad (16)$$

Using equations (16) and (15) and substituting them in equation (12), the dimensionless energy equation in z direction is obtained as:

$$uT\frac{2h(\tilde{T}_w-\tilde{T}_m)}{r_oU}=\frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \qquad (17)$$

After substituting the corresponding dimensionless parameters (1) and the velocity solution (9) in Eq. (17) we have:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+2Nu(1-r^2+\frac{1}{2}\beta_v)T=0 \qquad (18)$$

Governing dimensionless constraints are:

$$at\ \ r=0\rightarrow\frac{\partial T}{\partial r}=0 \qquad (19\text{-a})$$

$$at\ \ r=1\rightarrow T_w=-\beta_t\left.\frac{\partial T}{\partial r}\right|_{r=1}.$$
$$\beta_t=\frac{2-F_t}{F_t}\frac{Kn}{Pr}\frac{2\gamma}{1+\gamma} \qquad (19\text{-b})$$

where $\beta_t$ is temperature jump coefficient. It is clear that Eq. (18) and the presented boundary conditions (Eq. (19)) are homogenous. Thus, Eq. (18) is an eigenvalue differential equation where $Nu$, as the unknown constant, is in its closed form of formulation. The amount of Nusselt number can be calculated by solving Eq. (18) considering the related boundary conditions (19). Since the governing equation and the boundary conditions (Eqs. (18) and (19)) are hemogeneous, to calculate the non-zero dimensionless temperature distribution, another constraint is needed. One can find a physical constraint by performing a surface integration of the production of the dimensionless velocity profile ($u = \tilde{u}/U$) and the temperature distribution ($T = (\tilde{T} - \tilde{T}_w)/(\tilde{T}_m - \tilde{T}_w)$) on the cross section:

$$\int_A wTdA = \pi \quad (20)$$

In order to obtain an exact analytical solution for Eq. (18), the following form of the solution for $T$ is considered [19]:

$$T = \frac{f(\eta)}{r} \quad (21)$$

Here,

$$\eta = ar^2 \quad (22)$$

Where, $a$is an unknown constant. Thus, the following relations can be written for temperature derivatives:

$$\frac{\partial \mathrm{T}}{\partial \mathrm{r}} = \frac{-1}{\mathrm{r}^2}\mathrm{f} + 2\mathrm{a}\frac{\mathrm{df}}{\mathrm{d\eta}} \quad (23)$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{2}{r^3}f - \frac{2a}{r}\frac{df}{d\eta} + 4a^2 r\frac{d^2 f}{d\eta^2} \quad (24)$$

after substituting Eq. (24) and (23) in Eq. (18) we have:

$$\left(\frac{1}{4\eta^2} + \frac{\alpha Nu}{2a\eta} - \frac{Nu}{2a^2}\right)f = 0 \quad (25)$$

where $\alpha = 1 + \frac{1}{2}\beta_v$.

By considering the last unknown term of Eq. (18) to be equal to 1/4, the Whittaker differential equation is found. In other words, we may consider the unknown constant of this equation equal to $a = \sqrt{2Nu}$ to obtain the Whittaker differential equation from:

$$\frac{d^2 f}{d\eta^2} + \left(-\frac{1}{4} + \frac{\sqrt{2Nu}}{4}\frac{\alpha}{\eta} + \frac{1}{4\eta^2}\right)f = 0 \quad (26)$$

So the Whittaker differential equation can be written as follows [33-34]:

$$y'' + \left(-\frac{1}{4} + \frac{\mu}{z} + \frac{\frac{1}{4} - \nu^2}{z^2}\right)y = 0 \quad (27)$$

The solution for the Whittaker differential equation (Eq. (23)) is [34-35]:

$$y(x) = c_1 M_{\mu.\nu}(x) + c_2 W_{\mu.\nu}(x) \quad (28)$$

The following solution is valid for Eq. (29):

$$f(\eta) = C_1\, M_{\frac{\sqrt{2Nu}}{4}.0}(\eta) + C_2\, W_{\frac{\sqrt{2Nu}}{4}.0}(\eta) \quad (29)$$

where $\eta = \sqrt{2Nu}r^2$. The temperature distribution is obtained by substituting the Eq. (21) into the Eq. (29) as follows:

$$T = \frac{C_1}{r}M_{\frac{\alpha\sqrt{2Nu}}{4}.0}\left(\sqrt{2Nu}r^2\right) + \frac{C_2}{r}W_{\frac{\alpha\sqrt{2Nu}}{4}.0}\left(\sqrt{2Nu}r^2\right) \quad (30)$$

It is important to mention that the second term of Eq. (30) is singular at *r=0*. To avoid the singularity situation, it is necessary to consider$C_2 = 0$. The first term of Eq. (30) is not singular because the value of $M_{a,0}(br)$ tends to the $\kappa r$ when *r* approaches to zero.

So the temperature solution reduces to:

$$T = \frac{C_1}{r}M_{\frac{\alpha\sqrt{2Nu}}{4}.0}\left(\sqrt{2Nu}r^2\right) \quad (31)$$

In the presented form, there are two unknown, $C_1$and$Nu$, which should be calculated using the boundary conditions. The Nusselt number of flow inside the straight pipe is calculated using Eq. (19-b) as:

$$M_{\frac{1}{4}\alpha\sqrt{2Nu}.0}\left(\sqrt{2Nu}\right) + \beta_t(-M_{\frac{1}{4}\alpha\sqrt{2Nu}.0}\left(\sqrt{2Nu}\right) + 2((\frac{1}{2} - \frac{1}{4}\alpha)M_{\frac{1}{4}\alpha\sqrt{2Nu}.0}\left(\sqrt{2Nu}\right) + \frac{\sqrt{2}}{2}\frac{\left(\frac{1}{2} + \frac{1}{4}\alpha\sqrt{2Nu}\right)}{\sqrt{Nu}}M_{\frac{1}{4}\alpha\sqrt{2Nu}.0}\left(\sqrt{2Nu}\right) \quad (32)$$

$$(\sqrt{Nu}) = 0$$

The amount of first constant of Eq. (30) ($C_1$) is also determined by substituting Eqs. (31) into the Eq. (20).

## 4. Validation results

Fig. 3 shows the comparison between the closed form solution for fully developed temperature distribution due to forced convection in a micropipe in slip-flow regime and the exact solution presented in this paper. As it can be seen, there is a good agreement between current analytical solution and Hooman [36] solution. Although the results are obtained for the microscale problems, they can be

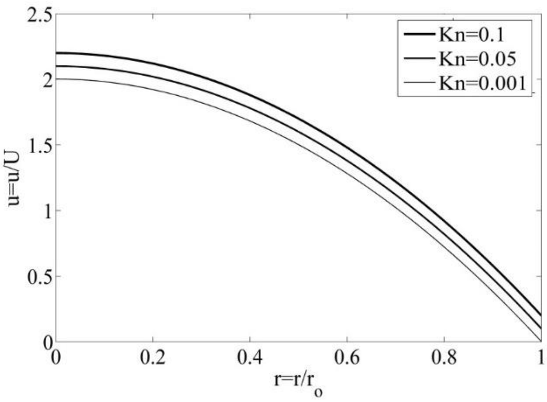


**Figure 2.** Velocity distribution in microtube for different Knudsen numbers.

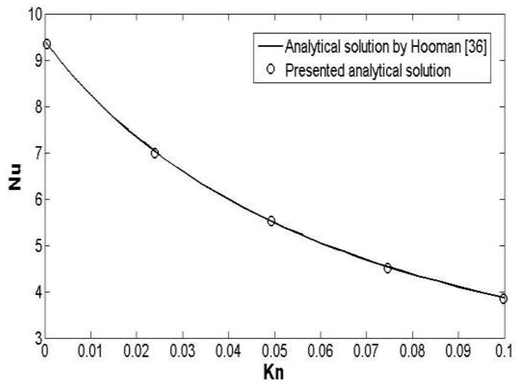


**Figure 3.** Comparison between the exact solution presented in this paper and the reported results by Hooman [36] for a micropipe in slip-flow regime.

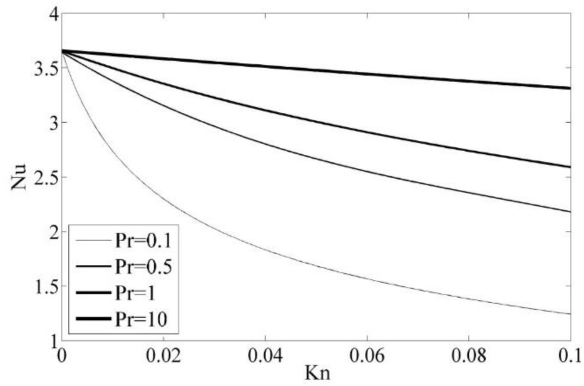


**Figure 4.** Variation of Physical Nusselt number versus Knudsen number for different Prandtl number.

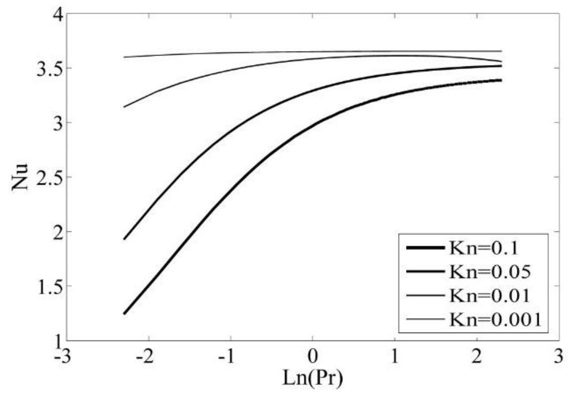


**Figure 5.** Variation of Physical Nusselt number versus Prandtl number for different Knudsen numbers.

generalized to the macroscale problems by considering $Kn = 0$

## 5. Results and discussion

In this section, the accuracy of the presented analytical solution is checked and an investigation on the effects of the slip flow and the temperature jump on the flow field and the heat transfers of gaseous fluids passing through micro scale pipes are presented. The flow distribution for the different Knudsen number is presented in Fig. 2. Data show that an increase in the Knudsen number generally leads to increment in both the maximum value of the velocity at the center of the pipe and the velocity of the fluid in the region near the wall. In following, the Nusselt number for different slip conditions are calculated and checked with previous studies. For this reason, the left hand side of the equation (32) is defined as:

$$F(Nu) = M_{\frac{1}{4}\alpha\sqrt{2Nu},0}\left(\sqrt{2Nu}\right) +\beta_t\Big(-M_{\frac{1}{4}\alpha\sqrt{2Nu},0}\left(\sqrt{2Nu}\right) +2\Big(\big(\frac{1}{2}-\frac{1}{4}\alpha\big)M_{\frac{1}{4}\alpha\sqrt{2Nu},0}\left(\sqrt{2Nu}\right) +\frac{\sqrt{2}}{2}\frac{\left(\frac{1}{2}+\frac{1}{4}\alpha\sqrt{2Nu}\right)}{\sqrt{Nu}}M_{\frac{1}{4}\alpha\sqrt{2Nu},0}\left(\sqrt{2Nu}\right)\Big)\sqrt{Nu}\Big) \quad (33)$$

In accordance with Eq. (33), variation of $F(Nu)$ versus the Nusselt number (Nusselts numbers are roots of Eq. (33)) for different Knudsen numbers and the Prandtl numbers are shown in Table 1 and 2. As these tables show, there are infinite number of discrete roots which are marked as circle symbols in tables. According to the scaling law, the Nusselt number of flow in a straight pipe is in order one [32-33]. Here, the second root is the only root which is in order one (For example *Nu= 3.6567856* in no-slip condition). Therefore, this root is corresponding to the physical solution and the other roots are considered as the mathematical solutions with no physical meaning.

This value in no-slip cases is exactly equal to the Nusselt number reported in the previous studies [32-33]. Fig. 4 shows the variation of this function in different employed

gaseous fluids and shows that an increment in the Knudsen number leads to a decrement in the heat transfer and the Nusselt number in isothermal cases. In Fig. 5, the physical Nusselt number variation against the Knudsen number in the different Prandtl number is plotted. It reveals that by increasing the Knudsen number the heat transfer coefficient generally decreases. Fig. 5 also shows the increment in the Prandtl number has the contrary effect and leads into an elevation in the Nusselt number values.

To achieve the complete form of the temperature distribution related to the other slip scenario, we must implement another constraint to calculate the corresponding possible eigenvalues. Due to the limited value of the temperature distribution at the center of pipe the $C_2$ constant must be considered as zero. Using constraint that is presented in Eq. (20) and substituting the achieved Nusselt numbers the $C_1$ obtains. In a non- slip scenario, by considering the value of 3.6567856 for the Nusselt number, the value of this constant calculates as $C_1 = 1.09615$. Some of other values of $C_1$ constant for the pre-assumed viscous fluid materials are collected in tables 1 and 2.

**Table 1.** The coefficient and the Nusselt number of possible solutions obtained from model analysis of temperature distribution for the different Knudsen number.

| Mode No. | Kn=0 (Noslip flow) | | Kn=0.001 | | Kn=0.01 | | Kn=0.1 | |
|---|---|---|---|---|---|---|---|---|
| | Nu | C | Nu | C | Nu | C | Nu | C |
| 1 (physical solution) | 3.65 | 1.09 | 3.64 | 1.07 | 3.47 | 1.08 | 2.45 | 0.94 |
| 2 | 22.30 | -3.19 | 22.26 | -3.19 | 21.90 | -3.18 | 18.77 | -3.11 |
| 3 | 56.96 | 5.54 | 56.85 | 5.54 | 56.00 | 5.55 | 48.63 | 5.76 |
| 4 | 107.62 | -8.04 | 107.45 | -8.04 | 105.90 | -8.08 | 92.69 | -8.85 |

**Table 2.** The coefficient and the Nusselt number of possible solutions obtained from modal analysis of temperature distribution for different Prandtl number at Kn=0.01 .

| Mode No. | Kn=0 (Noslip flow) | | Kn=0.001 | | Kn=0.01 | | Kn=0.1 | |
|---|---|---|---|---|---|---|---|---|
| | Nu | C | Nu | C | Nu | C | Nu | C |
| 1 (physical solution) | 3.65 | 1.09 | 3.64 | 1.07 | 3.47 | 1.08 | 2.45 | 0.94 |
| 2 | 22.30 | -3.19 | 22.26 | -3.19 | 21.90 | -3.18 | 18.77 | -3.11 |
| 3 | 56.96 | 5.54 | 56.85 | 5.54 | 56.00 | 5.55 | 48.63 | 5.76 |
| 4 | 107.62 | -8.04 | 107.45 | -8.04 | 105.90 | -8.08 | 92.69 | -8.85 |

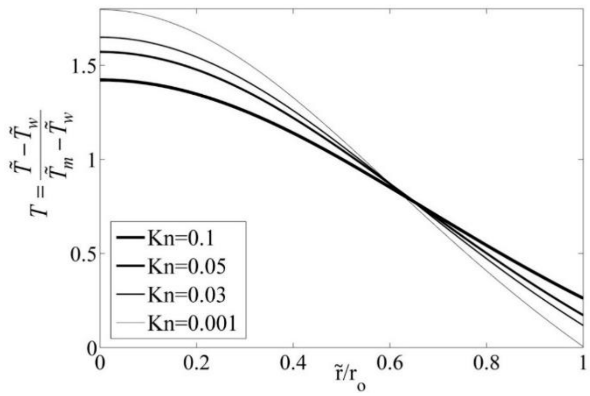


**Figure 6.** Variation of temperature for different Knudsen number at Prandtl =1 for physical Nusselt numbers.

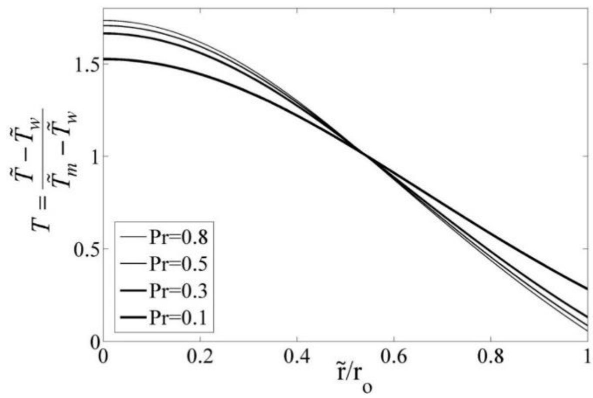


**Figure 7.** Variation of temperature for different Knudsen number at Prandtl =1 for physical Nusselt numbers.

For example, in No-Slip and Non-Temperature jump situation around the wall (Kn=0) the dimensionless temperature of flow inside the straight pipe under the constant wall temperature is derived as:

$$T=\frac{1.09615}{r}M_{0.6761,0}(2.7044r^2) \quad (34)$$

As mentioned previously, the Nusselt number in plain pipes is in order 1 which is indicated as mode 1 in this investigation.

After substituting appropriated Nusselt number values corresponding to slip scenarios and the calculated $C_1$ constant (presented in table 1 and 2), into Eq. (31), the dimensionless temperature of the flow inside the straight pipe under the constant wall temperature in micro-channels for different gaseous fluids will be derived. The temperature distributions achieve base in this solution for the different Knudsen numbers presented in Figs. 6 and 7.

## Conclusion

An exact analytical solution for the temperature

distribution and an analytical approach to calculate the heat transfer for the gaseous flow in the isothermal pipe is presented using the scaling law of the Nusselt number. The solution is derived in the form of *m* kind of Whittaker function using a constraint that has presented in previous parts. It is believed that the presented method can be used to derive exact analytical solutions of other similar scenarios such as the convective heat transfer in isothermal conduits with different geometries (such as curved or coiled ducts with circular or non-circular shape of cross section).

According to the presented analytical solution the principal conclusions of current studies are:

- It shows that employment of gaseous flow with the lower Knudsen numbers leads to higher heat transfers. Among the different properties of material and micro-channel situations, maximum heat transfer is related to the case that the Knudsen number is equal to zero (No-slip flow) for which the Nusselt number achieves its maximum value (*3.6567856)*.
- The Prandtl number exhibits the contrary effect on the variation of the Nusselt number and increments in this value can elevate the value of Nusslet number up to its maximum value around 3.6567856.

## Nomenclature

| | |
|---|---|
| $\tilde{r}$ | radial direction [m] |
| $r_0$ | radius of cross section [m] |
| $\tilde{z}$ | axial direction [m] |
| $G$ | Pressure gradient in main direction [Pa/m] |
| $h$ | Convection coefficient [$W\ /m^2K$ ] |
| $k$ | coefficient of conductivity [ $W\ /mK$ ] |
| $Nu$ | Nusselt number ( $Nu = 2hr_0/k$ ) |
| $\lambda$ | molecular mean free path |
| $Kn$ | Knudsen number ($\mathrm{Kn} = \lambda/\mathrm{r_o}$) |
| $F_t$ | the energy accommodation coefficient |
| $\gamma$ | the ratio of specific heats ($\mathrm{c_p}/\mathrm{c_v}$) |
| $\beta_t$ | dimensionless temperature jump coefficient ($\beta_t = 2\gamma\mathrm{Kn}(\mathrm{F_t}-2)/(1+\gamma)\mathrm{Pr}\mathrm{F_t}$) |
| $\tilde{p}$ | pressure [Pa] |
| $\tilde{T}$ | temperature [K] |
| $\tilde{u}$ | axial velocity component [m/s] |
| $U$ | bulk velocity [m/s] |
| $\eta$ | Viscosity [Pa s] |
| $\rho$ | Density[ $Kg\ /m^3$] |
| $\tilde{\tau}$ | Stress [Pa] |
| $F$ | the tangential momentum accommodation coefficient |
| $\beta_v$ | dimensionless |

## Appendix

### Whittaker function

A linear homogeneous ordinary differential equation of the second order:

$$\mathrm{w}'' + \left(-\frac{1}{4} + \frac{\mu}{\mathrm{x}} + \frac{(1/4) - \upsilon^2}{\mathrm{x}^2}\right)\mathrm{w} = 0 \qquad \text{(A1)}$$

where the variables$x, w$ and the parameters $\upsilon.\mu$may take arbitrary complex values. Equation (A1) represents the reduced form of a degenerate hypergeometric equation. For $\upsilon = 0$the Whittaker equation is equivalent to the Bessel equation.

The function $w_{\mu,\upsilon}(x)$satisfies the equation

$$w_{\mu.\upsilon}(x) = \frac{\Gamma(-2\upsilon)}{\frac{1}{2} - \mu - \upsilon} M_{\mu.\upsilon}(x) + \frac{\Gamma(2\upsilon)}{\frac{1}{2} - \mu + \upsilon} M_{\mu.-\upsilon}(x) \qquad \text{(A2)}$$

The pairs of functions$M_{\mu.\upsilon}(x).\, M_{\mu.-\upsilon}(x).$ and $W_{\mu.\upsilon}(x).\, W_{\mu.-\upsilon}(x)$ are linearly independent solutions of the equation (A1). The point $x = 0$is a branching point for$M_{\mu.\upsilon}(x)$, and $x = \infty$ is an essential singularity.

Relation with other functions:

with the degenerate hypergeometric function:

$$M_{\mu,\upsilon}(x) = x^{\upsilon+\frac{1}{2}} e^{\frac{-x}{2}} \Phi(\upsilon - \mu + \frac{1}{2}; 2\upsilon + 1; x) \qquad \text{(A3)}$$

with the modified Bessel functions and the Macdonald function:

$$M_{0.\upsilon}(x) = 2^{2\upsilon}\Gamma(\upsilon + 1)\sqrt{z} I_\upsilon(\frac{x}{2}) \qquad \text{(A4)}$$

$$W_{0.\upsilon}(x) = \sqrt{\frac{x}{\pi}} K_\upsilon(\frac{x}{2}) \qquad \text{(A5)}$$

with the probability integral:

$$W_{-\frac{1}{4}.-\frac{1}{4}}(x) = 2z^{\frac{1}{4}} e^{\frac{x}{2}} Erfc(\sqrt{x}) \qquad \text{(A6)}$$

with the Laguerre polynomials:

$$W_{n+\upsilon+\frac{1}{2}.\upsilon}(x) = n!\,(-1)^n z^{\upsilon+\frac{1}{2}} e^{\frac{-x}{2}} L_n{}^{2\upsilon}(x) \qquad \text{(A7)}$$

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